Episode 10

Angular Momentum

ENGN0040: Dynamics and Vibrations
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Topics for todays class

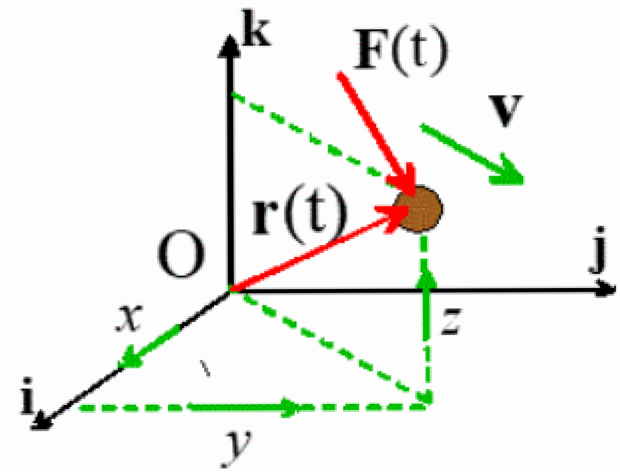
Angular Impulse-Momentum relations for particles

1. Definitions of angular impulse and angular momentum
2. Angular impulse-momentum relations for a single particle
3. Examples
4. Angular impulse-momentum relations for a system of particles
5. Example

4.6 Angular Impulse - Angular Momentum Relations for Particles

4.6.1 Definitions of Angular Impulse & Momentum

Moment of $\underline{F}$ about O $\quad \underline{M} = \underline{r} \times \underline{F}$



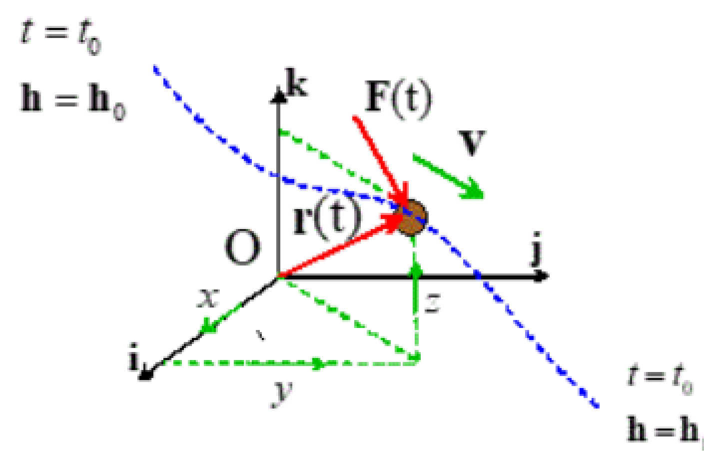
Angular Impulse about O

$$\underline{A} = \int_{t_0}^{t_1} \underline{M}(t)\,dt$$

Angular Momentum about O

$$\underline{h} = \underline{r} \times m\underline{v}$$

4.6.2 Angular Impulse – Momentum relation for a single particle



Version 1 :

$$\underline{M} = \underline{r} \times \underline{F} = \frac{d\underline{h}}{dt}$$

Version 2

$$\underline{A} = \underline{h}_1 - \underline{h}_0$$

Special Case $\underline{A} = \underline{0}$

$\underline{h}_1 = \underline{h}_0$ "Angular Momentum Conserved"

Proof

① $\underline{F} = m\underline{a} = m\frac{d\underline{v}}{dt}$

② $\underline{r} \times \underline{F} = \underline{r} \times m\frac{d\underline{v}}{dt}$

③ Note $\frac{d}{dt}\underbrace{(\underline{r} \times m\underline{v})}_{=\underline{h}} = \overbrace{\frac{d\underline{r}}{dt} \times m\underline{v}}^{=\underline{v} \times m\underline{v} = \underline{0}} + \underline{r} \times m\frac{d\underline{v}}{dt}$

Hence $\underline{r} \times F = \frac{d\underline{h}}{dt}$

④ Separate variables & integrate

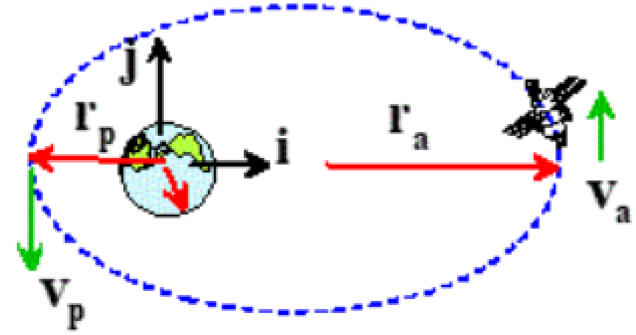
$$\underline{A} = \underline{h}_1 - \underline{h}_0$$

4.6.3: Example: The Ariane V launch vehicle puts satellites in a Geostationary Transfer Orbit (GTO) with

Perigee Altitude 250 km

Apogee Altitude 35950 km

The earth's radius is 6378 km. Calculate the speeds of the satellite at apogee and perigee



Apogee: furthest from earth

Perigee: closest to earth

Note: at these points $\underline{r}$ and $\underline{v}$ are orthogonal

Approach:

Two unknowns ⇒ need two equations

① Energy

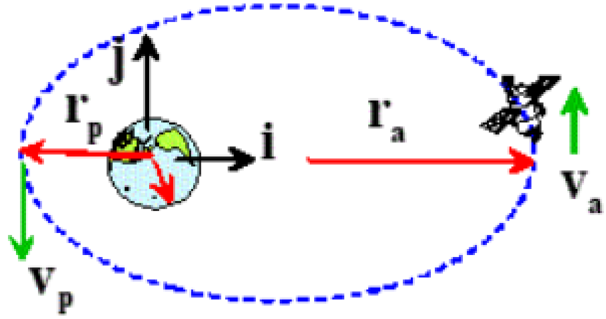
② Angular momentum

Energy

System = earth + satellite

No ext forces

$$\Rightarrow T_a + U_a = T_p + U_p$$



Let $\bar{V}_a$, $\bar{V}_p$ be speeds @ apogee & perigee

$$\frac{1}{2} m \bar{V}_a^2 - \frac{GMm}{r_a} = \frac{1}{2} m \bar{V}_p^2 - \frac{GMm}{r_p} \qquad (1)$$

Angular Momentum

Gravity on satellite acts towards O

$$\Rightarrow \underline{r} \times \underline{F}_{grav} = \underline{0}$$

Hence $\underline{h}_a = \underline{h}_p$

$$r_a \underline{i} \times m \bar{V}_a \underline{j} = -r_p \underline{i} \times (-m \bar{V}_p \underline{j})$$

$$\Rightarrow r_a \bar{V}_a = r_p \bar{V}_p \qquad (2)$$

Solve for V_p

$$V_p^2 \left\{ 1 - \frac{r_p^2}{r_a^2} \right\} = 2GM \left\{ \frac{1}{r_p} - \frac{1}{r_a} \right\}$$

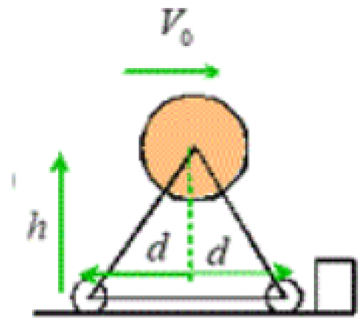
$$\Rightarrow \boxed{V_p = \sqrt{\frac{2GM}{(r_a + r_p)} \frac{r_a}{r_p}} = 10.2 \text{ km/sec}}$$

$$\boxed{V_a = \sqrt{\frac{2GM}{(r_a + r_p)} \frac{r_p}{r_a}} = 1.6 \text{ km/sec}}$$

Remark: GM is called "gravitational parameter"

For earth $GM = 3.986 \times 10^{14} \ m^3/sec^2$

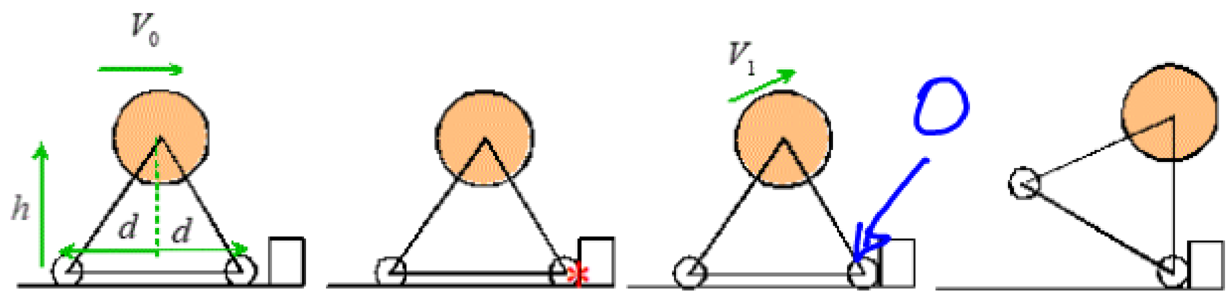
4.6.4: Example: A baby-walker has wheelbase $2d$ and its occupant's center of mass is a height h above the ground. Calculate the critical speed that will cause the walker to tip over if its front wheel strikes a stationary obstacle.



Goal: Interpret ASTM standard test for tipping resistance

Approach:

① "Baby" is stationary @ t_2 for critical speed



② For $t_1 < t < t_2$ earth + baby are conservative system ⇒ energy will give $\bar{V}_1$

③ For $t_1 < t < t_2$ circular motion about O

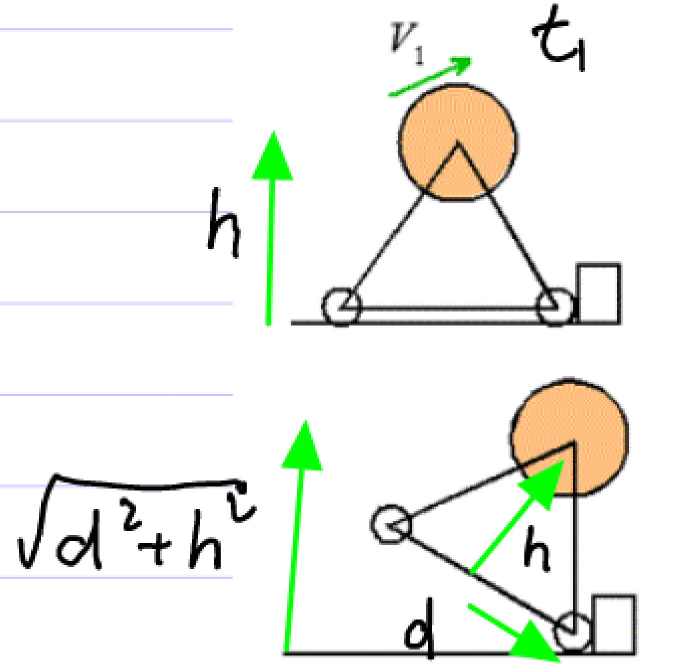
④ Impact force exerts no moment about O ⇒ angular momentum conserved during impact — will give $\bar{V}_0$

Energy

$$T_1 + U_1 = T_0 + U_0$$

$$\Rightarrow 0 + mg\sqrt{h^2+d^2} = \tfrac{1}{2} m \bar{V}_1^{\,2} + mgh$$

$$\Rightarrow \bar{V}_1 = \sqrt{2g}\left\{\sqrt{d^2+h^2} - h\right\}^{1/2}$$



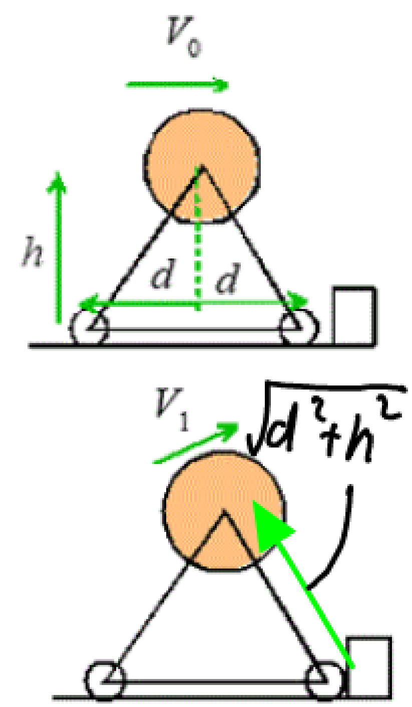
Angular Momentum

$$\underline{A} = \underline{0} \Rightarrow \underline{h}_1 = \underline{h}_0$$

$$\underline{h}_0 = -h\, m\, \bar{V}_0\, \underline{k}$$

$$\underline{h}_1 = -\sqrt{d^2+h^2}\; m\, \bar{V}_1\, \underline{k}$$

$$\Rightarrow \bar{V}_0 = \frac{\sqrt{d^2+h^2}}{h}\,\bar{V}_1$$

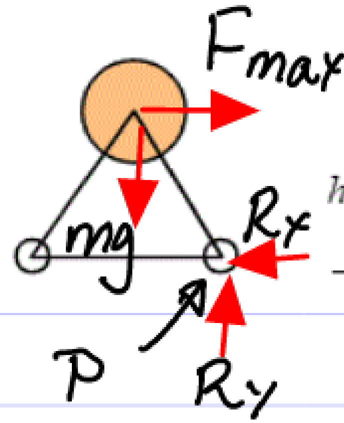


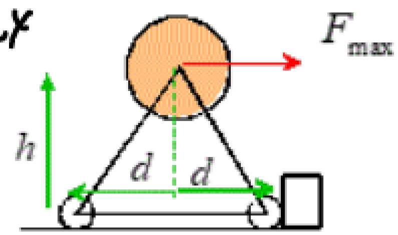
Combine:

$$\boxed{V_0 = \sqrt{2g}\,\frac{\sqrt{d^2+h^2}}{h}\left\{\sqrt{d^2+h^2}-h\right\}^{1/2}}$$

ASTM Stability Test

① Tipping Force Use Statics



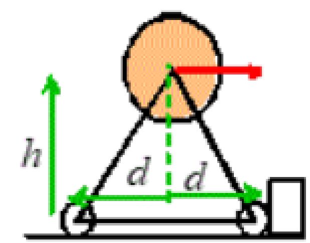


$$\sum \underline{M}_P = \underline{0}$$

$$\Rightarrow \quad (d\,mg - h\,F_{max})\,\underline{k} = \underline{0}$$

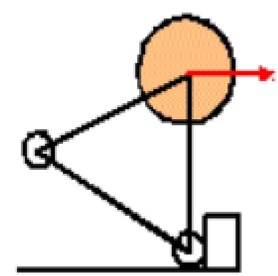
$$\Rightarrow \quad \boxed{F_{max} = mg\,d/h}$$

② Tipping Distance $= d$ (by inspection)



③ Stability Index $\underline{I} = F_{max}\,(lb) + d\,(in)$

$$\Rightarrow \quad I = mg\frac{d}{h} + d$$



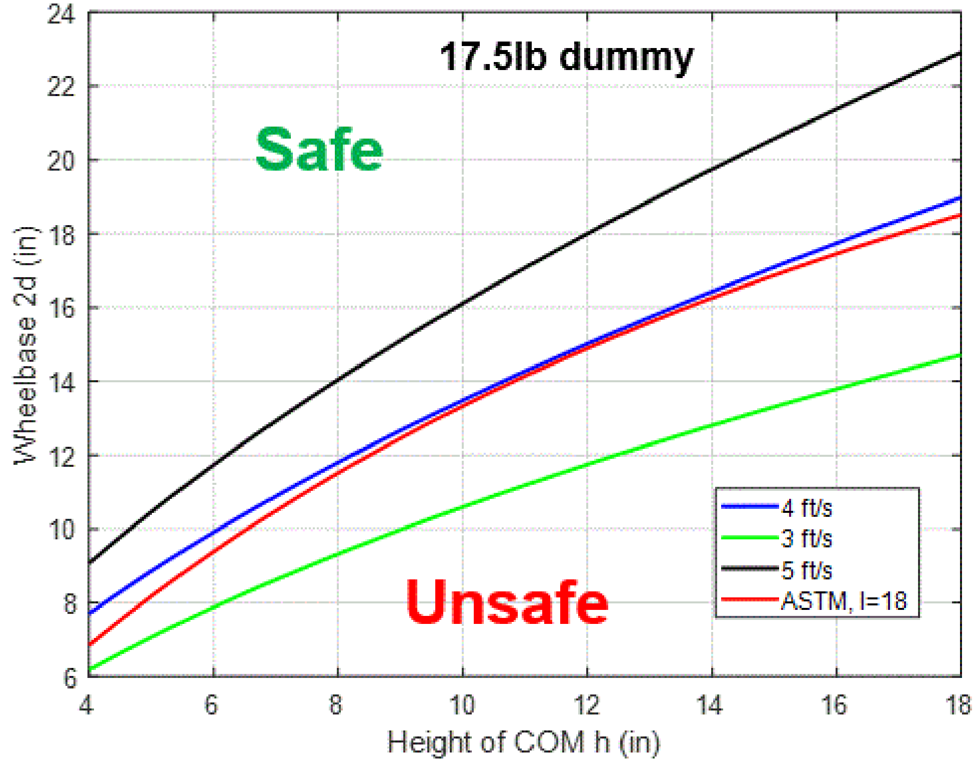
Comparison of prediction and ASTM standard

For a given height of COM h, we can find:

(1) Critical wheelbase $2d$ needed to pass ASTM test $2d = 2I/(1+mg/h)$

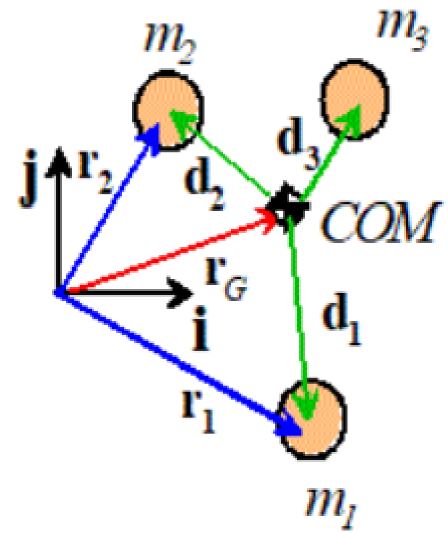
(2) Critical wheelbase to ensure walker will not tip for a given speed V_0 (solve numerically)

A walker that passes ASTM test has a tipping speed of approx 4 ft/s



4.6.3 Angular momentum of a system of particles

Angular momentum about O



Direct Summation

$$\underline{h}^{TOT} = \sum_{\text{particles}} \underline{r}_i \times m_i \underline{v}_i$$

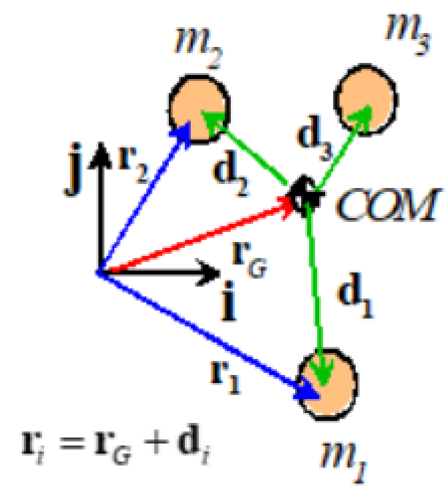
In terms of COM

Let $M = \sum m_i \qquad \underline{r}_G = \frac{1}{M} \sum m_i \underline{r}_i \qquad \underline{d}_i = \underline{r}_i - \underline{r}_G$

Version 1: $\underline{h}^{TOT} = \underline{r}_G \times M \underline{v}_G + \sum_i \underline{d}_i \times m_i \underline{v}_i$

Version 2: $\underline{h}^{TOT} = \underline{r}_G \times M \underline{v}_G + \sum_i \underline{d}_i \times m_i \underbrace{(\underline{v}_i - \underline{v}_G)}_{\text{velocity relative to COM}}$

Angular momentum in terms of COM



Preliminary results

(1) $$\mathbf{r}_G = \frac{1}{M}\sum_{Particles} m_i \mathbf{r}_i = \frac{1}{M}\sum_{Particles} m_i(\mathbf{r}_G + \mathbf{d}_i) = \mathbf{r}_G + \frac{1}{M}\sum_{Particles} m_i \mathbf{d}_i$$

$$\Rightarrow \sum_{Particles} m_i \mathbf{d}_i = \mathbf{0}$$

(2) $$\mathbf{p}^{TOT} = \sum_{Particles} m_i \mathbf{v}_i = M\mathbf{v}_G$$

Direct sum $$\mathbf{h}^{TOT} = \sum_{Particles} \mathbf{r}_i \times m_i \mathbf{v}_i$$

Hence $$\mathbf{h}^{TOT} = \sum_{Particles} (\mathbf{r}_G + \mathbf{d}_i) \times m_i \mathbf{v}_i = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} \mathbf{d}_i \times m_i \mathbf{v}_i$$ Version 1

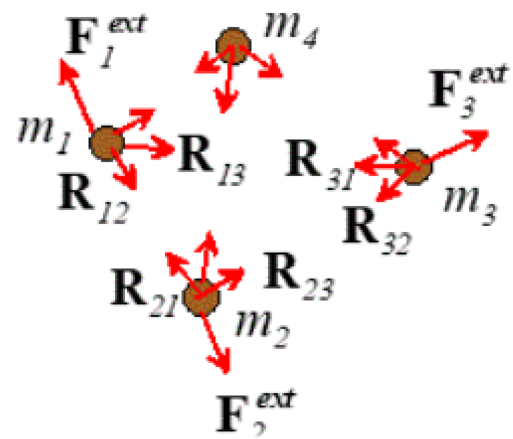
$$\mathbf{h}^{TOT} = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} \mathbf{d}_i \times m_i(\mathbf{v}_i - \mathbf{v}_G + \mathbf{v}_G)$$

$$\Rightarrow \mathbf{h}^{TOT} = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} \mathbf{d}_i \times m_i(\mathbf{v}_i - \mathbf{v}_G) + \left(\sum_{Particles} \mathbf{d}_i m_i\right) \times \mathbf{v}_G$$

$$\Rightarrow \mathbf{h}^{TOT} = \mathbf{r}_G \times M\mathbf{v}_G + \sum_{Particles} \mathbf{d}_i \times m_i(\mathbf{v}_i - \mathbf{v}_G)$$ Version 2

4·6·6 Angular Impulse-Momentum relations for a system of particles

$\mathbf{R}_{ij}$	Force exerted on particle i by particle j
$\mathbf{F}_i^{ext}$	External force on particle i
$\mathbf{v}_i$	Velocity of particle i



Assumption:

$\mathbf{R}_{ij}$ is parallel to $\mathbf{r}_j - \mathbf{r}_i$, ie

$$\mathbf{R}_{ij} = R_{ij}\frac{(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}} \qquad r_{ij} = |\mathbf{r}_j - \mathbf{r}_i|$$

Define:

(1) Total external moment $\quad \mathbf{M}^{TOT} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext}$

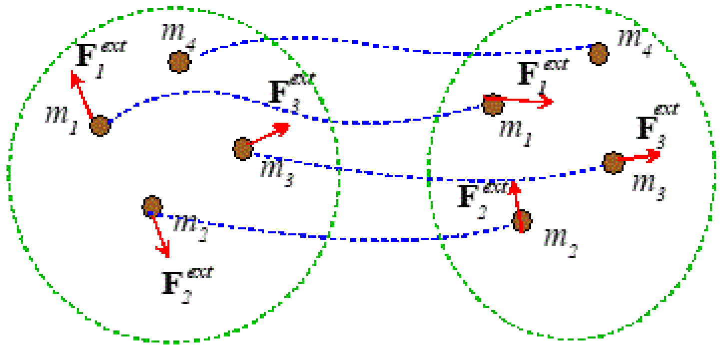
(2) Total external angular impulse

$$\mathbf{A}^{TOT} = \int_{t_0}^{t_1} \mathbf{M}(t)\,dt$$

Impulse - Momentum Formulas

Total External Moment $\mathbf{M}^{TOT}(t)$

Total External Angular Impulse $\mathbf{A}^{TOT} = \int_{t0}^{t1} \mathbf{M}^{TOT}(t)dt$



$t = t_0$ Total angular momentum $\mathbf{h}_0^{TOT}$

$t = t_1$ Total angular momentum $\mathbf{h}_1^{TOT}$

Version 1

$$\underline{M}^{TOT} = \frac{d\underline{h}^{TOT}}{dt}$$

Version 2

$$\underline{A}^{TOT} = \underline{h}_1^{TOT} - \underline{h}_0^{TOT}$$

Special Case

$$\underline{A}^{TOT} = \underline{0}$$

$$\underline{h}_1^{TOT} = \underline{h}_0^{TOT}$$

"Angular momentum is conserved"

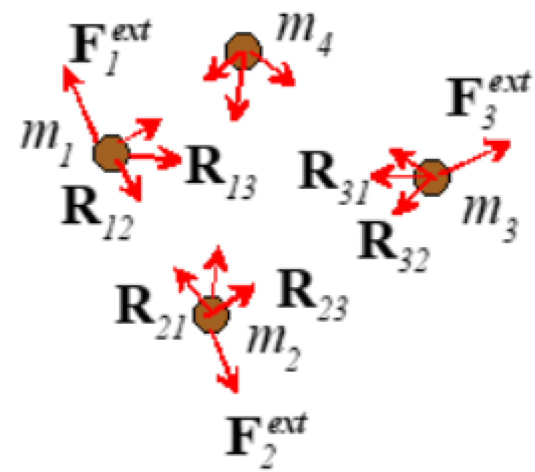
Proof of angular impulse-momentum relation for a system of particles

$\mathbf{R}_{ij}$ Force exerted on particle i by particle j

$\mathbf{F}_i^{ext}$ External force on particle i

$\mathbf{v}_i$ Velocity of particle i

Assume $\mathbf{R}_{ij} = R_{ij}\dfrac{(\mathbf{r}_j - \mathbf{r}_i)}{L_{ij}} \quad L_{ij} = |\mathbf{r}_j - \mathbf{r}_i|$



For one particle $\mathbf{r}_i \times \mathbf{F}_i^{ext} + \mathbf{r}_i \times \left(\sum_{Particles\ j} \mathbf{R}_{ij} \right) = \dfrac{d\mathbf{h}_i}{dt}$

Sum over all particles $\sum_{Forces} \mathbf{r}_i \times \mathbf{F}_i^{ext} + \sum_{Particles\ i} \sum_{Particles\ j} \mathbf{r}_i \times R_{ij} \dfrac{(\mathbf{r}_j - \mathbf{r}_i)}{L_{ij}} = \sum_{Particles\ i} \dfrac{d\mathbf{h}_i}{dt}$

Note
$$\left.\begin{array}{l} \mathbf{r}_i \times \mathbf{r}_i = \mathbf{0} \\ \mathbf{r}_i \times \mathbf{r}_j = -\mathbf{r}_j \times \mathbf{r}_i \qquad R_{ij} = R_{ji} \quad L_{ij} = L_{ji} \end{array}\right\} \Rightarrow \sum_{Particles\ i} \sum_{Particles\ j} \mathbf{r}_i \times R_{ij} \frac{(\mathbf{r}_j - \mathbf{r}_i)}{L_{ij}} = \mathbf{0}$$

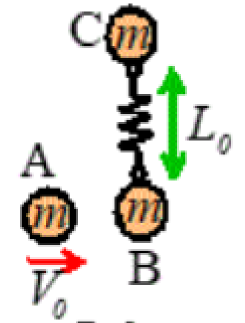
$$\boxed{\mathbf{M}^{TOT} = \frac{d\mathbf{h}^{TOT}}{dt}}$$

Separate variables and integrate

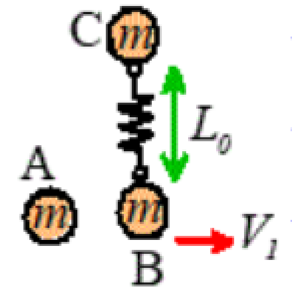
$$\boxed{\mathbf{A}^{TOT} = \mathbf{h}_1^{TOT} - \mathbf{h}_0^{TOT}}$$

4.6.7: Example: The two atoms in a diatomic molecule have mass m, and the bond between them has a potential energy

$$U(L) = E_0 \left\{ \left(\frac{L_0}{L} \right)^{12} - 2 \left(\frac{L_0}{L} \right)^{6} \right\}$$

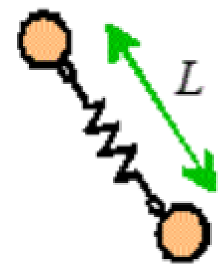


Before impact



Just after impact

At time t=0 the bond has length L_0 and the molecule is at rest. One of the atoms is then struck by an ion with mass m moving with speed v_0 in a direction perpendicular to the bond. The collision is elastic. Find a formula for the maximum subsequent length L of the bond. Hence, find a formula for the value of v_0 that will break the bond.



Max bond length

General Observations

(1) Bond exerts no force during impact

(2) Elastic collision $\Rightarrow$ A & B switch speeds

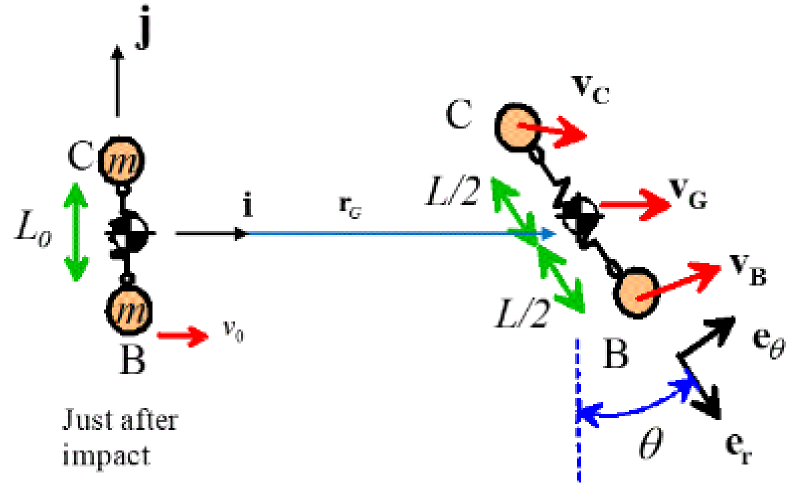
$\Rightarrow$ B has velocity $v_0 \underline{i}$ after collision

(3) No ext force on molecule after collision

$\Rightarrow$ energy, linear & angular momentum conserved

(4) at max bond stretch $\frac{dL}{dt} = 0$

Preliminaries: use polar coord formulas to find $\underline{v}_B$, $\underline{v}_C$



$$\underline{v}_B = \underline{v}_G + \underbrace{\frac{d}{dt}\left(\frac{L}{2}\right)}_{=0} \underline{e}_r + \frac{L}{2}\frac{d\theta}{dt}\,\underline{e}_\theta$$

$$\underline{v}_C = \underline{v}_G - \frac{L}{2}\frac{d\theta}{dt}\,\underline{e}_\theta \quad \text{(by symmetry)}$$

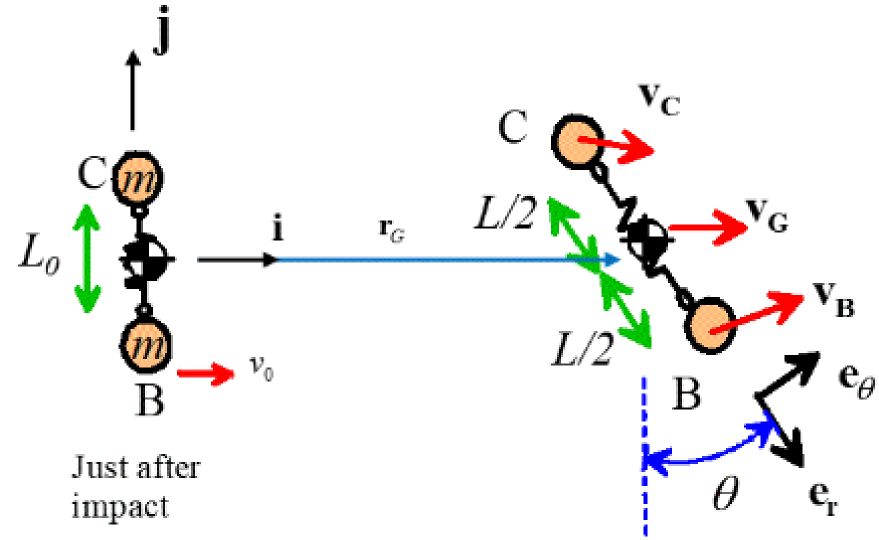
Linear momentum

$$\underline{p}_0^{TOT} = m v_0 \underline{i} \qquad \underline{p}_1^{TOT} = M \underline{v}_G = 2m\,\underline{v}_G$$

$$\underline{p}_1^{TOT} = \underline{p}_0^{TOT} \Rightarrow \qquad \boxed{\underline{v}_G = \frac{v_0}{2}\underline{i}}$$

Also note $\underline{r}_G = \underline{v}_G t = \frac{v_0}{2} t\, \underline{i}$

Preliminaries : use polar coord formulas for $\underline{v}_B$, $\underline{v}_c$



$$\underline{V}_B = \underline{V}_G + \underbrace{\cancel{\frac{dL}{dt}}}_{=0} \underset{\sim}{e_r} + \frac{L}{2}\frac{d\theta}{dt}\,\underline{e_\theta}$$

$$\underline{V}_c = \underline{V}_G - \frac{L}{2}\frac{d\theta}{dt}\,\underline{e_\theta}$$

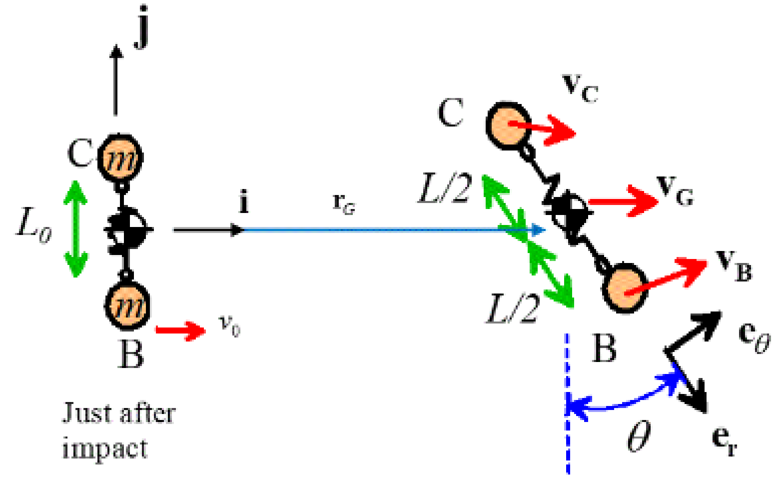
Linear momentum

$$p_0^{TOT} = m V_0 \underline{i} \qquad p_1^{TOT} = \underline{M}\,\underline{V}_G = 2m\,\underline{V}_G$$

$$p_1^{TOT} = p_0^{TOT} \Rightarrow \boxed{\underline{V}_G = \frac{V_0}{2}\underline{i}}$$

also $\boxed{\underline{r}_G = \underline{V}_G\, t}$

Angular momentum



$$\underline{h}_0 = -\frac{L_0}{2}\,\underline{j} \times m v_0 \underline{i} = \frac{L_0}{2} m v_0 \underline{k}$$

$$\underline{h}_1 = \underbrace{\underline{r}_G \times M \underline{v}_G}_{=\underline{0} \text{ since } \underline{r}_G = \underline{v}_G t} + \sum_i m_i \underline{d}_i \times \underline{v}_i$$

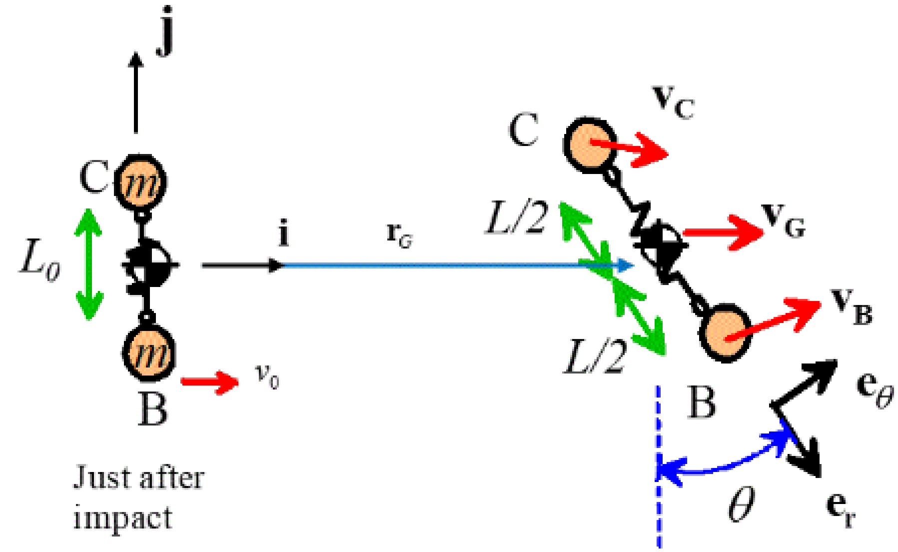
$$\Rightarrow \underline{h}_1 = m \frac{L}{2} \underline{e}_r \times \left(\underline{v}_G + \frac{L}{2}\frac{d\theta}{dt} \underline{e}_\theta \right)$$

$$+ m \left(-\frac{L}{2} \underline{e}_r \right) \times \left(\underline{v}_G - \frac{L}{2}\frac{d\theta}{dt} \right) \underline{e}_\theta$$

$$\underline{e}_r \times \underline{e}_\theta = \underline{k}$$

$$\Rightarrow \underline{h}_1 = m \frac{L^2}{2} \frac{d\theta}{dt} \underline{k}$$

$$\underline{h}_1 = \underline{h}_0 \Rightarrow \boxed{\frac{L}{2}\frac{d\theta}{dt} = \frac{v_0}{2}\frac{L_0}{L}}$$



Energy $\quad T_1 + U_1 = T_0 + U_0$

$$T_0 = \frac{1}{2} m v_0^2 \qquad U_0 = -E_0$$

$$T_1 = \frac{1}{2} m |\underline{v}_B|^2 + \frac{1}{2} m |\underline{v}_C|^2$$

$$U_1 = E_0 \left\{ \left(\frac{L_0}{L}\right)^{12} - 2\left(\frac{L_0}{L}\right)^6 \right.$$

Note:

$$|\underline{v}_B|^2 = \underline{v}_B \cdot \underline{v}_B = \underline{v}_G \cdot \underline{v}_G + L\frac{d\theta}{dt}\underline{e}_\theta \cdot \underline{v}_G + \left(\frac{L}{2}\frac{d\theta}{dt}\right)^2$$

$$|\underline{v}_C|^2 = \underline{v}_C \cdot \underline{v}_C = \underline{v}_G \cdot \underline{v}_G - L\frac{d\theta}{dt}\underline{e}_\theta \cdot \underline{v}_G + \left(\frac{L}{2}\frac{d\theta}{dt}\right)^2$$

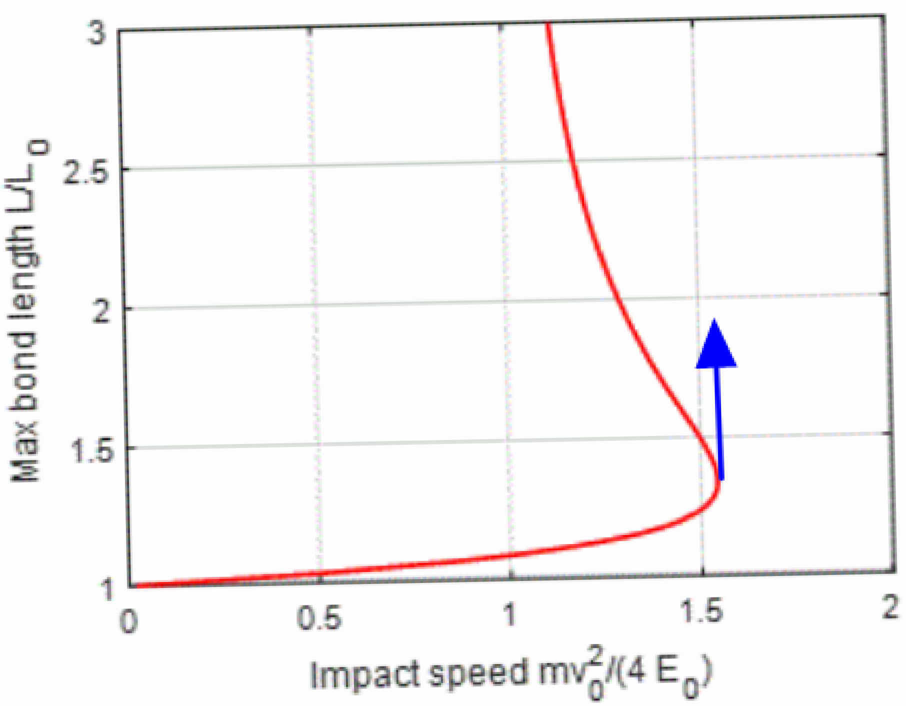
$$\Rightarrow T_1 = m\left(\overbrace{|\underline{v}_G|^2}^{=v_0^2/4} + \left(\overbrace{\frac{L}{2}\frac{d\theta}{dt}}^{=(v_0/2)(L_0/L)}\right)^2\right) = \frac{m v_0^2}{4}\left(1 + \frac{L_0^2}{L^2}\right)$$

Hence $T_1 + U_1 = T_0 + U_0 \Rightarrow$

$$\frac{mV_0^2}{4}\left(1 + \frac{L_0^2}{L^2}\right) + E_0\left\{\left(\frac{L_0}{L}\right)^{12} - 2\left(\frac{L_0}{L}\right)^{6}\right\} = \frac{mV_0^2}{2} - E_0$$

$$\Rightarrow \frac{mV_0^2}{4E_0} = \frac{(L_0/L)^{12} - 2(L_0/L)^6 + 1}{1 - (L_0/L)^2}$$

We can plot this



Note $L \to \infty$ for $\frac{mV_0^2}{4E_0} > 1.54$

Critical speed to break bond

$$\boxed{V_0 = 2.49\sqrt{\frac{E_0}{m}}}$$